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Resonant amplification of gravity waves over a circular sill

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The trapping of gravity waves over a circular sill, originally studied by Longuet-Higgins (1967), is re-examined in the limit $\delta \equiv d_1/d \downarrow 0$ with $\lambda \equiv \sigma^2 d/g = O(1)$, where d_1 is the depth over the sill, d the outer depth, and σ the angular frequency of the incident wave. Explicit results are obtained for the resonance curves. These results are in qualitative agreement with the corresponding shallow-water approximations ($\lambda \ll 1$) of Longuet-Higgins, which have been questioned by Renardy (1983). A remarkably simple result is obtained for the mean-square response to a broadband, randomly phased incident wave.

1. Introduction

Following Longuet-Higgins (1967), Pite (1977) and Renardy (1983), I consider the motion induced over a circular sill by a gravity wave incident from a laterally unbounded ocean. (The corresponding scattering problem has been treated by Black, Mei & Bray (1971), although they did not consider the strong resonances that are of primary concern here.) The basic similarity parameters are

$$\alpha \equiv \frac{a}{d_1}, \quad \delta \equiv \frac{d_1}{d}, \quad \lambda \equiv \frac{\sigma^2 d}{g}, \tag{1.1a,b,c}$$

where (see figure 1) a is the radius of the sill, d_1 is the depth over the sill, d is the depth of the outer ocean, and σ is the angular frequency of the incident wave.

The wave motion is resonantly amplified over the sill if $\delta \ll 1$ and $k_1 a$ approximates j_{mn}, one of the doubly infinite, discrete set of positive zeros of the Bessel functions J_m, where the wavenumber k_1 is determined by

$$k_1 d_1 \tanh k_1 d_1 = \delta\lambda. \tag{1.2}$$

The corresponding reciprocal Q (to which the radiation damping is proportional) has the form [see (4.6)]

$$\frac{1}{Q} = \delta F(ka, kd, k_1 d_1), \tag{1.3}$$

where the wavenumber k is determined by

$$kd \tanh kd = \lambda. \tag{1.4}$$

The corresponding measure of viscous damping in the boundary layers at the free surface and over the surface of the sill is given by (Miles 1967*a*)

$$\frac{1}{Q} = \frac{\dot{E}}{\sigma E} = k_1 \left(\frac{\nu}{2\sigma}\right)^{\frac{1}{2}} \left(\frac{C + \operatorname{sech}^2 k_1 d_1}{\tanh k_1 d_1}\right), \tag{1.5}$$

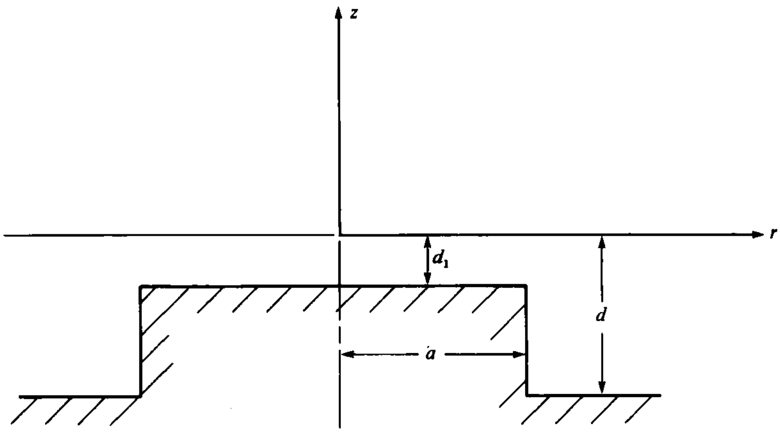

FIGURE 1. Cross-section of circular sill in laterally unbounded ocean of depth d.

where $\dot{E}$ is the mean dissipation rate per unit area, E is the energy per unit area (integrated over a vertical column), ν is the kinematic viscosity, and C is a parameter that varies from 0 for a clean free surface through a maximum of 2 for a contaminated surface to an asymptotic value of 1 for an inextensible film (representative values for the open sea and a laboratory tank are 0 and 1 respectively). The kinematic viscosity may be replaced by an eddy viscosity in the coefficient of $\operatorname{sech}^2 k_1 d_1$ if the boundary layer over the sill is turbulent. The reciprocal Q given by (1.5) must be added to that given by (1.3) to obtain the total damping for each mode.

Longuet-Higgins (1967) assumes $\lambda \ll 1$ and invokes continuity of pressure and mass flux across the annular aperture (the projection of the cylindrical boundary from the sill to the free surface) to match an interior superposition of standing waves to an exterior superposition of incident and reflected waves. This procedure, which is exact in the limit $\lambda \downarrow 0$ with α and δ fixed, neglects the interior and exterior families of non-propagated modes that are excited by the geometrical discontinuity at, and fall off exponentially from, the aperture. Longuet-Higgins also considers the effects of the Earth's rotation (significant for very long gravity waves), viscous damping [obtaining the counterpart of (1.5) above for $C = 0$ and $k_1 d_1 \ll 1$], an incident pulse, and a broadband, randomly phased wave. His approximations appear to be quantitatively adequate for very long waves in the real ocean, but not for typical laboratory configurations.

Pite (1977) does not impose the restriction $\lambda \ll 1$ and incorporates an artificial damping of Rayleigh's type (specific force proportional to velocity, which is compatible with irrotational motion). He, too, neglects the non-propagated modes (but without the rational justification provided by $\lambda \ll 1$) and matches the propagating modes through continuity of pressure and energy flux (rather than mass flux). He obtains moderately good agreement between laboratory measurements and his analytical calculations using an experimentally inferred friction coefficient. This agreement notwithstanding, the introduction of Rayleigh-type friction does not appear to offer any advantage over, and is less realistic than, the conventional procedure of calculating boundary-layer dissipation as a small perturbation on the inviscid flow (cf. Longuet-Higgins 1967 and Miles 1967a). Nor does Pite's somewhat arbitrary procedure of using the full dispersion relation (1.4) but neglecting the non-propagated modes appear to offer any advantage over the straightforward solution of the scattering problem followed by simplification for $\delta \ll 1$ or other parametric restrictions.

Renardy (1983), apparently unaware of the antecedent work of Black *et al.* (1971), examines the full linear problem and concludes that Longuet-Higgins's approximate solution may yield rather misleading results. It is clear that the shallow-water approximation may be quantitatively inadequate if kd is not small, but Renardy concludes that it may be qualitatively wrong in predicting strong amplification for each of the aforementioned resonances if $\delta \ll 1$. It appears to me, however, that Renardy's conclusion is based on an inappropriate comparison and on an erroneous order-of-magnitude estimate for an infinite series (her X). The resonant frequencies of the interior waves, say $\tilde{\sigma}_{mn}$, differ from those determined by the aforementioned zeros of the Bessel functions, say σ_{mn}, by $O(\delta)$† in the limit $\delta \downarrow 0$. The difference $\tilde{\sigma}_{mn} - \sigma_{mn}$ comprises a contribution from the corresponding radiated mode together with contributions from the non-propagated modes; Longuet-Higgins's calculation allows for the former but not for the latter, whereas Renardy's calculation (as well as the present calculation) allows for both. The contributions of the interior and exterior non-propagated modes are proportional to δX and δY respectively, in Renardy's notation, and she alleges that $X = O(1/\delta^2)$ and $Y = O(1)$. In fact, $X \equiv O(\ln \delta)$ (see Appendix A), and her numerical estimates of $\tilde{\sigma}_{mn}$ appear to differ from those of Longuet-Higgins only by $O(\delta)$. But this difference is of the same order as the width of the resonance curve, which is measured by (1.3) above, and the calculation of the resonant peaks using Longuet-Higgins's approximations to $\tilde{\sigma}_{mn}$ in Renardy's formulation is manifestly misleading.

Experiments designed to test the above theoretical predictions have been carried out by Pite (see above) and Barnard, Pritchard & Provis (1983), who find 'no evidence...to suggest the occurrence of the dramatic resonances predicted theoretically' and conclude that 'The reason why there is such a large disparity between [their] experimental results and the theoretical model is not understood'. They do, however, suggest several possible causes for the disparity, and Pritchard recently (private communication) has conjectured that the most likely explanation is that the wave response is 'dominated by reflections between the island and the wavemaker'.

Against this background, it seemed to me to be worthwhile to re-examine the limit $\delta \downarrow 0$ and to obtain explicit, analytical representations of Q and $\tilde{\sigma}_{mn} - \sigma_{mn}$ in this limit. I give the basic formulation in §2, starting from the assumption of a monochromatic incident wave in a homogeneous, inviscid liquid. In §3, I construct interior (over the sill) and exterior solutions through separation of variables. These solutions comprise the exterior disturbance for a vertical cylinder that penetrates the free surface augmented by additional disturbances that may be regarded as excited by the (unknown) radial velocity in the aperture. The continuity of the velocity potential across the aperture provides an integral equation for this aperture velocity, which may be expanded in the normal modes of the sill to obtain an infinite set of linear equations for the expansion coefficients. I find that this set may be diagonalized in the limit $\delta \downarrow 0$ to obtain the surface-wave amplitude over the sill within an error factor of $1 + O(\delta)$. This provides first approximations to $\tilde{\sigma}_{mn} - \sigma_{mn}$ and $1/Q$, which I exploit in §5 to obtain a uniformly valid approximation to the resonance curve. I find that, in this approximation, the non-propagated modes make no contribution to Q and that only the exterior non-propagated modes contribute to $\tilde{\sigma}_{mn} - \sigma_{mn}$. (I differ from Renardy in the latter conclusion.)

† $O(\delta)$ implicitly includes $O(\delta \ln \delta)$.

In most problems of oceanographic interest the incident wave is random and represented by power and directional spectra, which describe the energy distribution over frequency and angle of incidence, and it then is the integral of the square of the complex amplitude over each of the resonant peaks that is of primary interest. I consider this problem in §6 and obtain the remarkably simple result [see (6.4)]

$$\overline{\zeta^2} = \frac{2}{\delta} \sum_{m=0}^{\infty} (2-\delta_{0m}) \sum_{n=1}^{\infty} \frac{\sigma_{mn} P(\sigma_{mn})}{j_{mn}^2} C(kd, k_1 d_1) \tag{1.6}$$

for the mean-square displacement (averaged over both time and the sill) of the free surface, where P is the power spectrum of the incident wave, defined such that

$$\overline{\zeta_{\mathrm{inc}}^2} = \int_0^\infty P(\sigma)\,\mathrm{d}\sigma, \tag{1.7}$$

j_{mn} is the nth positive zero of J_m, and C is a depth-dependent factor that reduces to 1 in the limit $\lambda \downarrow 0$.

I conclude that Longuet-Higgins's formulation and results are qualitatively valid, at least for $\delta \ll 1$, which is the domain of principal interest (since resonant amplification is otherwise small).

2. Formulation

The monochromatic incident wave is described by the real part of

$$\zeta = \zeta_0 \,\mathrm{e}^{\mathrm{i}(kr\cos\theta - \sigma t)} \tag{2.1a}$$

$$= \zeta_0 \,\mathrm{e}^{-\mathrm{i}\sigma t} \sum_{m=0}^{\infty} \epsilon_m \,\mathrm{i}^m J_m(kr) \cos m\theta \quad (\epsilon_m \equiv 2-\delta_{0m}), \tag{2.1b}$$

where ζ is the complex displacement of the free surface, r, θ and z are cylindrical polar coordinates (see figure 1), ζ_0 is the complex amplitude of ζ at the origin, σ is the angular frequency, δ_{0m} is the Kronecker delta, and J_m is a Bessel function. The corresponding complex velocity potential is determined by

$$\nabla^2\phi = 0 \quad \begin{pmatrix} 0 \leqslant r < a, & -d_1 < z < 0 \\ r > a, & -d < z < 0 \end{pmatrix}, \tag{2.2}$$

$$\phi_z = 0 \quad \begin{pmatrix} 0 \leqslant r < a, & z = -d_1 \\ r > a, & z = -d \end{pmatrix}, \tag{2.3}$$

$$\phi_r = 0 \quad (r = a, \quad -d < z < -d_1), \tag{2.4}$$

$$\phi_z = \zeta_t, \quad \phi_t + g\zeta = 0 \quad (z = 0), \tag{2.5a,b}$$

and the radiation condition

$$\zeta \sim \zeta_0 \,\mathrm{e}^{-\mathrm{i}\sigma t}\left\{\mathrm{e}^{\mathrm{i}kr\cos\theta} + \left(\frac{a}{r}\right)^{\frac{1}{2}} \mathrm{e}^{\mathrm{i}kr} A(\theta)\right\} \quad (kr \uparrow \infty), \tag{2.6}$$

where $A(\theta)$ is a dimensionless scattering amplitude (see Appendix B).

3. Separation of variables

Following Miles & Gilbert (1968) or Black *et al.* (1971), we pose the solution of (2.2)–(2.6) in the form [cf. (2.1 *b*)]

$$\phi = (\mathrm{i}\sigma)^{-1} g\zeta_0 \,\mathrm{e}^{-\mathrm{i}\sigma t} \sum_{m=0}^{\infty} \epsilon_m \,\mathrm{i}^m \Phi_m(r,z) \cos m\theta, \tag{3.1 a}$$

$$\zeta = \zeta_0 \,\mathrm{e}^{-\mathrm{i}\sigma t} \sum_{m=0}^{\infty} \epsilon_m \,\mathrm{i}^m \Phi_m(r,0) \cos m\theta, \tag{3.1 b}$$

where
$$\Phi_m = \sum_{\kappa_1} A_\kappa^m G_m^{(1)}(\kappa r)\, Z_\kappa^{(1)}(z) \quad (0 < r < a), \tag{3.2 a}$$

$$\Phi_m = F_m(kr)\frac{Z_2(z)}{Z_2(0)} + \sum_{\kappa_2} B_\kappa^m G_m^{(2)}(\kappa r)\, Z_\kappa^{(2)}(z) \quad (r > a), \tag{3.2 b}$$

$$F_m(kr) = \frac{J_m(kr)\,H_m'(ka) - J_m'(ka)\,H_m(kr)}{H_m'(ka)}, \tag{3.3}$$

J_m and $H_m \equiv H_m^{(1)}$ are Bessel and Hankel functions of the first kind,

$$G_m^{(1)}(\kappa r) = \frac{I_m(\kappa r)}{\kappa a I_m'(\kappa a)}, \quad G_m^{(2)}(\kappa r) = \frac{K_m(\kappa r)}{\kappa a K_m'(\kappa a)}, \tag{3.4 a, b}$$

I_m and K_m are modified Bessel functions of the first and second kind,

$$Z_\kappa^{(n)}(z) = 2^{\frac{1}{2}}\left[1 + \frac{\sin 2\kappa d_n}{2\kappa d_n}\right]^{-\frac{1}{2}} \cos\kappa(z+d_n) \quad (n = 1,2), \tag{3.5}$$

$d_2 \equiv d$, and the summations in (3.2) are over the eigenvalues determined by

$$\kappa_n d_n \tan \kappa_n d_n = -\left(\frac{\sigma^2}{g}\right) d_n \equiv -\lambda_n \quad (n = 1,2). \tag{3.6}$$

The $Z_\kappa^{(n)}$ constitute a complete, orthonormal set in $-d_n < z < 0$. The eigenvalues comprise an infinite discrete set of positive real members and a single imaginary member, $\kappa_n \equiv -\mathrm{i}k_n$ ($k_2 \equiv k$), for which (3.4)–(3.6) transform to

$$G_m^{(1)}(-\mathrm{i}k_1 r) = \frac{J_m(k_1 r)}{k_1 a J_m'(k_1 a)}, \quad G_m^{(2)}(-\mathrm{i}kr) = \frac{H_m(kr)}{ka H_m'(ka)}, \tag{3.7 a, b}$$

$$Z_n \equiv Z_{-\mathrm{i}k_n}^{(n)} = 2^{\frac{1}{2}}\left[1 + \frac{\sinh 2k_n d_n}{2k_n d_n}\right]^{-\frac{1}{2}} \cosh k_n(z+d_n) \quad (n = 1,2) \tag{3.8}$$

and
$$k_n d_n \tanh k_n d_n = \lambda_n \quad (n = 1,2). \tag{3.9}$$

(The negative real eigenvalues and the positive imaginary eigenvalue $\mathrm{i}k_1$ are redundant; the positive imaginary eigenvalue $\mathrm{i}k_2$ is ruled out by the radiation condition.)

Those modes for which κ is real decay exponentially with $|r-a|$, by virtue of which the solution in the interior domain is dominated by the oscillatory mode described by (3.7 *a*) and (3.8), and (3.1 *b*), may be approximated by

$$\zeta \sim \zeta_0 \,\mathrm{e}^{-\mathrm{i}\sigma t} \sum_{m=0}^{\infty} \epsilon_m \,\mathrm{i}^m A_1^m G_m^{(1)}(-\mathrm{i}k_1 r)\, Z_1(0) \cos m\theta \quad (A_1^m \equiv A_{-\mathrm{i}k_1}^m). \tag{3.10}$$

The corresponding approximation to the mean-square displacement, averaged over both the sill and time, is given by

$$\frac{\overline{\zeta^2}}{\frac{1}{2}|\zeta_0|^2} = \sum_{m=0}^{\infty} \epsilon_m \left|\frac{A_1^m}{k_1 a}\right|^2 [1+(k_1^2 a^2 - m^2)\, G_m^2(k_1 a)]\,[Z_1(0)]^2, \tag{3.11}$$

where $\frac{1}{2}|\zeta_0|^2$ is the mean-square of the incident wave (2.1), and

$$G_m(k_1 a) \equiv G_m^{(1)}(-\mathrm{i}k_1 a) = \frac{J_m(k_1 a)}{k_1 a J'_m(k_1 a)}. \tag{3.12}$$

The first term on the right-hand side of (3.2*b*) represents the solution for a circular cylinder that penetrates the free surface, for which $d_1 \equiv 0$ and (2.2), (2.3) and (2.5*a*, *b*) hold only in $r > a$. The remaining terms in (3.2) may be regarded as driven by the radial velocity in the annular aperture, which we represent by

$$u_m(z) \equiv a\left(\frac{\partial \Phi_m}{\partial r}\right)_{r=a} \quad (-d_1 < z < 0). \tag{3.13}$$

Substituting (3.2*a*) into (3.13) and invoking the orthogonality of the $Z_\kappa^{(1)}$, we obtain

$$u_m(z) = \sum_{\kappa_1} A_\kappa^m Z_\kappa^{(1)}(z), \quad A_\kappa^m = \frac{1}{d_1}\int_{-d_1}^{0} u_m(z)\, Z_\kappa^{(1)}(z)\,\mathrm{d}z \equiv \langle u_m Z_\kappa^{(1)}\rangle, \tag{3.14a, b}$$

where, here and subsequently, $\langle\ \rangle$ denotes a vertical average over the aperture. Substituting (3.2*b*) into (3.13) and augmenting the result with $\partial\Phi_m/\partial r = 0$ in $-d < z < -d_1$, which follows from (2.4), we obtain

$$B_\kappa^m = \frac{1}{d}\int_{-d_1}^{0} u_m(z)\, Z_\kappa^{(2)}(z) = \delta\langle u_m Z_\kappa^{(2)}\rangle \tag{3.15a}$$

$$= \delta \sum_{\kappa_1} A_\kappa^m \langle Z_\kappa^{(1)} Z_\kappa^{(2)}\rangle, \tag{3.15b}$$

where (3.15*b*) follows from (3.15*a*) through (3.14*a*). [The substitution of (3.14*b*) and (3.15*a*) into (3.16) yields an integral equation for $u_m(z)$, which provides the basis for a variational formulation of the scattering problem (Miles 1967*b*, 1971; Black *et al.* 1971).]

Invoking the continuity of ϕ, which in turn implies the continuity of Φ_m, across the aperture, we obtain

$$\sum_{\kappa_1} A_\kappa^m G_m^{(1)}(\kappa a)\, Z_\kappa^{(1)}(z) - \sum_{\kappa_2} B_\kappa^m G_m^{(2)}(\kappa a)\, Z_\kappa^{(2)}(z) = F_m(ka)\frac{Z_2(z)}{Z_2(0)} \quad (-d_1 < z < 0). \tag{3.16}$$

Multiplying (3.16) through by $Z_\mu^{(1)}(z)$, averaging the result over the aperture, and eliminating B_κ^m through (3.15*b*), we obtain the infinite matrix equation

$$[\delta_{\mu\nu} G_m^{(1)}(\mu a) + \delta C_{\mu\nu}^m]\{A_\nu^m\} = \{D_\mu^m\}, \tag{3.17}$$

where μ and ν span the complete set $\{\kappa_1\}$,

$$C_{\mu\nu}^m = -\sum_{\kappa_2} G_m^{(2)}(\kappa a)\langle Z_\mu^{(1)} Z_\kappa^{(2)}\rangle\langle Z_\nu^{(1)} Z_\kappa^{(2)}\rangle, \tag{3.18}$$

and

$$D_\mu^m = F_m(ka)\frac{\langle Z_\mu^{(1)} Z_2\rangle}{Z_2(0)}. \tag{3.19}$$

4. Narrow-aperture approximation ($\delta \ll 1$)

A uniformly valid first approximation to the solution of (3.17) for $\delta \ll 1$ may be obtained by neglecting $\delta C^m_{\mu\nu}$ except in the resonant neighbourhoods defined by $G^{(1)}_m(-ik_1 a) = G_m(k_1 a) = O(\delta)$; in particular,

$$A^m_1 = \frac{D^m_1 + O(\delta)}{G_m(k_1 a) + \delta C^m_{11}} \quad (C^m_{11} \equiv C^m_{-ik_1,\,-ik_1}, \quad D^m_1 \equiv D^m_{-ik_1}). \tag{4.1}$$

Setting $\mu = -ik_1$ in (3.19) and invoking (3.3) and the Wronskian relation

$$J_m(x)\,H'_m(x) - J'_m(x)\,H_m(x) = i[J_m(x)\;Y'_m(x) - J'_m(x)\;Y_m(x)] = \frac{2i}{\pi x}, \tag{4.2}$$

we obtain
$$D^m_1 = F_m(ka)\frac{\langle Z_1 Z_2\rangle}{Z_2(0)}, \quad F_m(ka) = \frac{2i}{\pi ka H'_m(ka)}. \tag{4.3a,b}$$

Setting $\mu = \nu = -ik_1$ in (3.18) and simplifying the imaginary part of the result with the aid of (4.2) and (4.3b) (note that $G^{(2)}_m$ is real except for $\kappa = -ik$), we obtain

$$C^m_{11} = -\sum_{\kappa_2} G^{(2)}_m(\kappa a)\langle Z_1 Z^{(2)}_\kappa\rangle^2 = -R_m + \tfrac{1}{2}i(\delta Q)^{-1}, \tag{4.4}$$

where (see also Appendix A)

$$R_m = \operatorname{Re} G^{(2)}_m(-ika)\langle Z_1 Z_2\rangle^2 + \sum_{\kappa_2} G^{(2)}_m(\kappa a)\langle Z_1 Z^{(2)}_\kappa\rangle^2, \tag{4.5}$$

in which the summation now is over the real roots of (3.6), and

$$\frac{1}{Q} = \pi\delta\,|F_m(ka)|^2\langle Z_1 Z_2\rangle^2. \tag{4.6}$$

Substituting (4.4) into (4.1), we obtain

$$A^m_1 = \frac{D^m_1}{G_m(k_1 a) - \delta R_m + \frac{1}{2}iQ^{-1}}, \tag{4.7}$$

where, here and subsequently, an error factor of $1 + O(\delta)$ is implicit.

5. Resonant peaks

The resonant frequencies correspond to the zeros of the real part of the denominator of (4.7), which, in turn, correspond to the positive zeros of J_m in the limit $\delta \downarrow 0$. Expanding (3.12) about one of these zeros, $j_{mn} (0 < j_{m1} < j_{m2} < \ldots)$, we obtain

$$G_m(k_1 a) \doteqdot \frac{k_1 a - j_{mn}}{j_{mn}}, \tag{5.1}$$

which is, by definition, $O(\delta)$ in the neighbourhood of resonance. Substituting (5.1) into (4.7) and introducing

$$\sigma_{mn} \equiv \left[\left(\frac{g}{a}\right) j_{mn} \tanh\left(\frac{j_{mn}}{\alpha}\right)\right]^{\frac{1}{2}}, \quad \tilde{\sigma}_{mn} \equiv \sigma_{mn}(1 + \delta R_m), \tag{5.2a,b}$$

we obtain
$$A^m_1 = \frac{\sigma_{mn} D^m_1}{\sigma - \tilde{\sigma}_{mn} + \frac{1}{2}i(\sigma_{mn}/Q)} \quad \left[\frac{\sigma - \tilde{\sigma}_{mn}}{\sigma_{mn}} = O(\delta)\right]. \tag{5.3}$$

(m,n)	j_{mn}	$k_1 d_1$	$\tilde{\sigma}a/(gd)^{\frac{1}{2}}$ LH	(5.2)	A_{LH}	A_{R}	$A_{(5.5)}$	$Q_{(4.6)}$
2, 1	5.136	0.041	4.92	4.75	16.23	11.88	14.96	34.1
8, 1	12.225	0.098	12.12	11.95	3895	0.0797	2791	2.51×10^6

TABLE 1. The amplification factor A for $\delta = \frac{1}{16}$ and $\alpha = 125$, as calculated from Longuet-Higgins (1967), Renardy (1983) and (5.5) herein for two particular resonances. The values of $|H'_m|$ required in (5.5) were taken from Morse & Feshbach (1953).

We note that, within this resonant neighbourhood, k is determined by

$$k\tanh kd = \frac{\sigma^2_{mn}}{g} = \frac{j_{mn}}{a}\tanh\frac{j_{mn}}{\alpha}. \tag{5.4}$$

The resonant maximum of $|A^m_1|$ is $2Q|D^m_1|$. The corresponding maximum of Longuet-Higgins's (1967) amplification factor ($|A_n|$ in his notation, wherein n corresponds to m herein) is given by

$$A \equiv |A_m|_{\mathrm{LH}} = \frac{|A^m_1|}{k_1 aJ'_m(k_1 a)} = \frac{1}{\delta}\left|\frac{kaH'_m(ka)}{k_1 aJ'_m(k_1 a)}\right|[\langle Z_1 Z_2\rangle Z_2(0)]^{-1}, \tag{5.5}$$

in which $k_1 a$ may be approximated by j_{mn} and k is determined by (5.4).

The results given by (5.3), (5.5) and (4.6) for two particular cases are compared with Longuet-Higgins's approximation and Renardy's (1983) 'full linear theory' calculation in table 1. The differences between the Longuet-Higgins and present approximations for the (2, 1) mode are within the expected error factor for $1+O(\delta)$. The corresponding difference in A for the (8, 1) mode stems in part (a factor of 0.90) from the correction for finite depth but mainly from differences in the numerical calculations (I believe that the present method is more accurate if $Q \gg 1$; however, dissipation must be significant, especially at laboratory scales, in the determination of the actual Q for these higher modes). The difference between Longuet-Higgins's and Renardy's results appears to stem from the $O(\delta)$ differences in their respective approximations to the resonant frequencies (see discussion in §1).

6. Broadband excitation

The preceding results are for a monochromatic incident wave, whereas the spectrum of the incident disturbance is typically much wider than the width of the resonant peaks ($\sim \sigma_{mn}/Q$). Let $P(\sigma)$ and $D(\theta_i)$, where θ_i is the angle of incidence, be the power and directional spectra of the incident wave, normalized according to

$$\int_0^\infty P(\sigma)\,\mathrm{d}\sigma = \tfrac{1}{2}\zeta_0^2, \quad \int_{-\frac{1}{2}\pi}^{\frac{1}{2}\pi} D(\theta_i)\,\mathrm{d}\theta_i = 1. \tag{6.1a, b}$$

The mean-square displacement, averaged over both the sill and time on the assumption that the contributions of the resonant peaks dominate the contribution of the non-resonant part of the spectrum [the error factor associated with this approximation is $1+O(\delta)$], is given by [cf. (3.10) and (3.11)]

$$\overline{\zeta^2} = \int_0^\infty P(\sigma)\,d\sigma \int_{-\frac{1}{2}\pi}^{\frac{1}{2}\pi} D(\theta_i)\,d\theta_i \sum_{m=0}^{\infty} \frac{\epsilon_m^2 |A_1^m|^2 [Z_1(0)]^2}{\pi a^2} \int_0^a [G_m^{(1)}(-ik_1 r)]^2 r\,dr \times \int_0^{2\pi} \cos^2 m(\theta-\theta_i)\,d\theta \quad (6.2a)$$

$$= \sum_{m=0}^{\infty} \epsilon_m \sum_{n=1}^{\infty} \frac{P(\sigma_{mn})}{j_{mn}^2} [Z_1(0)]^2 \int_0^\infty |A_1^m|^2\,d\sigma, \quad (6.2b)$$

where (6.2b) follows from (6.2a) after carrying out the integrations with respect to r, θ and θ_i, invoking (6.1b), letting $k_1 a = j_{mn}$, summing over n, and invoking $G_1(j_{mn}) = 0$. Invoking (5.3), integrating over the resonant peak(s), and substituting (4.3a) and (4.6) into the result, we obtain

$$\int_0^\infty |A_1^m|^2\,d\sigma = 2\pi Q \sigma_{mn} |D_1^m|^2 = \frac{2\sigma_{mn}}{\delta[Z_2(0)]^2} \quad (6.3)$$

within $1+O(\delta)$. Substituting (6.3) into (6.2b) and invoking (5.2a), we obtain

$$\overline{\zeta^2} = \frac{2}{\delta} \sum_{m=0}^{\infty} \epsilon_m \sum_{n=1}^{\infty} \frac{\sigma_{mn} P(\sigma_{mn})}{j_{mn}^2} \left[\frac{Z_1(0)}{Z_2(0)}\right]^2, \quad (6.4)$$

wherein $k_1 \doteqdot j_{mn}/a$ and k is implicitly determined by (5.4) in $[Z_1(0)/Z_2(0)]$.

The incorporation of viscous damping requires the introduction of the factor $Q_\nu/(Q_\nu+Q_r)$ in (6.4), where Q_ν and Q_r are given by (1.5) and (4.6) respectively.

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Appendix A. Approximations to R_m

We consider approximations to the series in (4.5),

$$S_m \equiv \sum_{\kappa_2} G_m^{(2)}(\kappa a) \langle Z_1 Z_\kappa^{(2)} \rangle^2 \quad (A\,1)$$

where the summation is over the infinite, discrete set of real roots of (3.6) with $d_2 \equiv d$ therein.

We first suppose that

$$\lambda \ll 1, \quad \frac{a}{d} = \alpha\delta \gg 1. \quad (A\,2a,b)$$

It follows from (A 2a) that the real roots of (3.6) may be approximated by

$$\kappa = \left(\frac{n\pi}{d}\right)\left[1+O\left(\frac{\lambda}{n}\right)\right] \quad (n = 1,2,\ldots). \quad (A\,3)$$

Substituting (A 3) into (3.4b) and invoking (A 2b) and the asymptotic approximation to K_m, we obtain

$$G_m^{(2)}(\kappa a) \sim -(m^2+\kappa^2 a^2)^{-\frac{1}{2}} \quad (A\,4a)$$

$$\sim -\frac{d}{n\pi a} \quad \left(\frac{n\pi a}{d} \gg m\right). \quad (A\,4b)$$

Substituting (A 3) into (3.5) and invoking $\delta \equiv d_1/d$, we obtain

$$\langle Z_1 Z_\kappa^{(2)} \rangle \simeq 2^{\frac{1}{2}}(-)^n (n\pi\delta)^{-1} \sin n\pi\delta. \tag{A 5}$$

Substituting (A 3), (A 4*b*) and (A 5) into (A 1), we obtain

$$S_m \sim -\left(\frac{d}{a}\right)\left(\frac{2}{\pi^3\delta^2}\right)\sum_{n=1}^{\infty}\frac{\sin^2 n\pi\delta}{n^3} \equiv -\left(\frac{d}{a}\right)X(\delta), \tag{A 6}$$

where X is the series that appears in Renardy's equation (4.5), which is equivalent to (4.7) herein, after invoking (A 2), if the series Y is neglected, or to Longuet-Higgins's equation (7.7) if both X and Y are neglected. Renardy alleges that $X = O(1/\delta^2)$ as $\delta \downarrow 0$; in fact, $X = O(\ln \delta)$, as is evident from the transformation [cf. Mangulis 1965, §3Ac(8)]

$$\pi X = \frac{1}{\pi^2\delta^2}\sum_{n=1}^{\infty}\frac{1-\cos 2n\pi\delta}{n^3} \tag{A 7a}$$

$$= \frac{1}{2\pi^2\delta^2}\int_0^\infty \left[\frac{1}{\mathrm{e}^t-1}-\left(\frac{\mathrm{e}^t\cos 2\pi\delta-1}{\mathrm{e}^{2t}-2\,\mathrm{e}^t\cos 2\pi\delta+1}\right)\right]t^2\,\mathrm{d}t \tag{A 7b}$$

$$= 2\int_0^\epsilon \frac{t\,\mathrm{d}t}{t^2+(2\pi\delta)^2}+\int_\epsilon^\infty \frac{\mathrm{e}^t(\mathrm{e}^t+1)\,t^2\,\mathrm{d}t}{(\mathrm{e}^t-1)^3}+O(\epsilon\ln\delta) \quad [\epsilon\downarrow 0,\quad \delta = o(\epsilon)] \tag{A 7c}$$

$$= -2\ln(2\pi\delta)+5+O(\delta^{\frac{2}{3}}\ln\delta) \quad (\epsilon = \delta^{\frac{2}{3}}). \tag{A 7d}$$

Now suppose that

$$\lambda \gg 1, \quad \lambda_1 \ll 1, \quad \alpha \gg 1. \tag{A 8a, b, c}$$

It follows from (A 8*a*) and (3.5) that

$$Z_\kappa^{(2)} \sim \left(\frac{2}{k^2+\kappa^2}\right)^{\frac{1}{2}}(\kappa\cos\kappa z+k\sin\kappa z), \tag{A 9}$$

the real-κ spectrum is continuous over $(0, \infty)$, and

$$\sum_{\kappa_2} f(\kappa) \sim \frac{d}{\pi}\int_0^\infty f(\kappa)\,\mathrm{d}\kappa, \tag{A 10}$$

where, here and subsequently, error factors of $1+O(1/\lambda)$ are implicit. The approximations in §4 remain valid, but with implicit error factors of $1+O(\lambda_1)$ rather than $1+O(\delta)$. Substituting

$$\langle Z_1 Z_\kappa^{(2)} \rangle \sim \left(\frac{2}{k^2+\kappa^2}\right)^{\frac{1}{2}}\left(\frac{\kappa}{k_1^2+\kappa^2}\right)\left(\frac{\kappa\sin\kappa d_1+k\cos\kappa d_1}{d_1}\right) \tag{A 11}$$

into (A 1), invoking (A 10), introducing the new variable $u = \kappa d_1$, invoking (A 4*a*) by virtue of (A 8*c*), and comparing with the corresponding result for a two-dimensional shelf [Miles 1967, equations (6.6) and (6.10) with $\beta = \lambda_1^{\frac{1}{2}} \ll 1$ therein], we obtain

$$\left(\frac{a}{d}\right)S_m \sim -\frac{2}{\pi}\int_0^\infty \frac{(u\sin u+\lambda_1\cos u)^2\,u^2\,\mathrm{d}u}{[u^2+(m/\alpha)^2]^{\frac{1}{2}}(u^2+\lambda_1^2)(u^2+\lambda_1)^2} \tag{A 12a}$$

$$= -\frac{2}{\pi}\left(1-\frac{m^2}{k_1^2a^2}\right)^{-\frac{1}{2}}\left[0.230+\ln\frac{1}{\lambda_1}+M\left(\frac{m}{ka}\right)\right], \tag{A 12b}$$

where

$$M(z) = \ln\frac{2}{z}-\tfrac{1}{2}(1-z^2)^{-\frac{1}{2}}\ln\left[\frac{1+(1-z^2)^{\frac{1}{2}}}{1-(1-z^2)^{\frac{1}{2}}}\right]. \tag{A 13}$$

We remark that $\tilde{\sigma}_{mn}-\sigma_{mn} = O(1/\alpha)$, rather than $O(\delta)$, in this approximation.

Appendix B. Scattering amplitude

Letting $kr \uparrow \infty$ in (3.2*b*), invoking

$$H_m(kr) \sim \mathrm{i}^{-m}(\tfrac{1}{2}\pi kr)^{-\frac{1}{2}}\, \mathrm{e}^{\mathrm{i}(kr-\frac{1}{4}\pi)} \quad (kr \uparrow \infty), \tag{B 1}$$

substituting into (3.1*b*), invoking (2.1), and comparing the result with (2.6), we obtain

$$A(\theta) = \mathrm{e}^{-\frac{1}{4}\mathrm{i}\pi}\left(\frac{2}{\pi ka}\right)^{\frac{1}{2}} \sum_{m=0}^{\infty} \frac{\epsilon_m}{H'_m(ka)}\left[\frac{B^m_{-1k} Z_2(0)}{ka} - J'_m(ka)\right]\cos m\theta. \tag{B 2}$$

It follows from (3.15*b*) that $B^m_{-1k} = O(\delta)$ except near resonance, where it may be approximated by

$$B^m_{-1k} = \delta A^m_1 \langle Z_1 Z_2\rangle [1+O(\delta)]. \tag{B 3}$$

The resonant peak value, obtained by substituting $A^m_1 = -2\mathrm{i}QD^m_1$ from (5.3) and invoking (4.3) and (4.6), is

$$B^m_{-1k} = H'^{*}_m(ka) \equiv J'_m(ka) - \mathrm{i}\,Y'_m(ka). \tag{B 4}$$

Thus, the square-bracketed term in (B 2) varies from $-J'_m(ka)$ at non-resonant points to $-\mathrm{i}\,Y'_m(ka)$ at resonant points.

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